Method for solving logical loops in system reliability analysis

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Abstract: The procedure for solving Boolean equations with unknown element(s) is presented. Discussions are made for operation of typical engineering systems with loop structure. It is revealed that components are necessary to be classified into three types. Time-dependent expression of a component's state is given and operating states of loop structures are identified by Boolean algebraic procedure. The procedure proposed in this paper is applicable to the condition that components can start at any time in system operational sequence, and each component has multiple chances to be started. A sample system was analyzed and the result was confirmed by a step by step analysis. The procedure shown in this paper is very useful in evaluating engineering systems which have logical loop structure(s), and also useful in effectively designing high reliable systems.

Keyword: system reliability; availability analysis; loop structure; boolean equation

1 Introduction

Main task of reliability engineering is to assess system performance. System performance can be expressed by Boolean relations among states of components or sub-systems in the system. Then, it is necessary to solve Boolean equations for the evaluation of system performance.

But, for a system which has logical loop structure(s), the Boolean relations have to be described with unknown variable(s). The number of unknown variables equals to the number of essential logical loop structures existed in the system. If we try to solve the Boolean equation(s) with unknown variable(s), we encounter infinite circulation of the unknown variable(s). Logical loop was not generally solved in terms of the arithmetic operators of Boolean algebra. Many attempts[1-5] have been proposed. An exact method[6] has been proposed for solving this problem in reliability analysis, but it was effective only in a restricted analysis condition, that is, in the condition that almost all the components are started at the same time.

In the present paper, a generalized procedure is proposed for determining the terms which represent the operating state of loop structure. The procedure is applicable to the condition that components can start at any time in system operational sequence, and each component has multiple chances to be started.

An example system is taken up and solved by this procedure with detailed explanations.

2 Solution of Boolean equation with unknown element

Now consider a simple system as shown in Fig. 1., which is a generalized system configuration for a single loop structure. The U is upstream part against loop structure. The D is placed downstream from loop. The components A and B are the elements of loop, and only A directly connects to D.

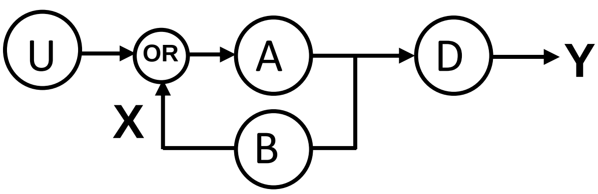

Fig. 1 Basic loop structure.

Italic characters *U, A, B, D* represent sets of operating states of components U, A, B, D, respectively. Italic characters *X, Y* represent the sets of events in which outputs of components B or D exists. The outputs of components A, B, D exist when the upstream components are in operating states. Then, following Boolean relations are obtained.

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$$Y = (U + X)AD, \qquad X = (U + X)AB. \tag{1}$$

Substitute the relation $X = (U+X)\,AB$ into the first equation,

$$Y = (U + (U + X)AB)AD. \tag{2}$$

Rearrange the equation (2), then,

$$Y = (U + XAB)AD = UAD + XABD. \tag{3}$$

Unknown element X cannot be eliminated, and the relation expressed by equation (3) does not change after repeating the substitution of X into equation (3). This is the situation that infinite circulation of unknown element appears in the process of solving a Boolean equation with unknown element(s).

Consider the next form of equation as Boolean equation with unknown element x.

$$x = f\left(\alpha_1,\cdots,\alpha_n\right)x + g\left(\alpha_1,\cdots,\alpha_n\right), \tag{4}$$

where α_i be independent Boolean variables (fixed elements of a Boolean algebra) and x be dependent Boolean variables (unknown elements of a Boolean algebra).

Equation G=H is equivalent to the following equation in Boolean algebra,

$$G\bar{H} + \bar{G}H = \phi. \tag{5}$$

Then, equation(4) becomes,

$$x\left(\bar{f} + \bar{x}\right)\bar{g} + \bar{x}\left(f\,x + g\right) = \phi. \tag{6}$$

It is transformed to the next equation,

$$x\,\bar{f}\,\bar{g} + \bar{x}\,g = \phi. \tag{7}$$

The Boolean theorems [7] say the Boolean equation $xA + \bar{x}B = \phi$ has a solution if and only if $AB=\phi$ has a solution, and the solution is $x = m\bar{A} + B.$
Where m is an arbitrary Boolean element.
The relation $AB=\phi$ is equivalent to $\bar{f}\,\bar{g}\,g = \phi.$

This is an identical equation. Therefore the solution of equation(4) becomes,

$$x = mf\left(\alpha_1,\cdot\cdot,\alpha_n\right) + g\left(\alpha_1,\cdot\cdot,\alpha_n\right). \tag{8}$$

If we decompose f into multiple terms, equation(4) becomes as follows,

$$x = \left\{f_1 + f_2 + \cdots + f_k\right\}x + g, \tag{9}$$

and the solution of equation(9) becomes,

$$x = m_1 f_1 + m_2 f_2 + \cdots + m_k f_k + g\;. \tag{10}$$

The unknown element x can be expressed by α_i, m (or m_k) without x.

3 Solutions of simultaneous Boolean equations with unknown elements

Consider the next form of equations as simultaneous Boolean equations,

$$x_i = f_i\left(\alpha_1,\cdot\cdot,x_1,\cdot\cdot,x_{i-1},x_{i+1},\cdot\cdot,x_n\right)x_i + g_i\left(\alpha_1,\cdot\cdot,x_1,\cdot\cdot,x_{i-1},x_{i+1},\cdot\cdot,x_n\right) \qquad i = 1\cdots n, \tag{11}$$

where α_i be independent Boolean variables (fixed elements of a Boolean algebra) and x_i be dependent Boolean variables (unknown elements of a Boolean algebra).

The procedure for solving simultaneous Boolean equations becomes as follows. Start from the first equation of equation (11).

$$x_1 = f_1\left(\alpha_1,\cdot\cdot,x_2,\cdots,x_n\right)x_1 + g_1\left(\alpha_1,\cdot\cdot,x_2,\cdots x_n\right) \tag{12}$$

The solution of equation (12) becomes

$$x_1 = m_1 f_1\left(\alpha_1,\cdot\cdot,x_2,\cdots,x_n\right) + g_1\left(\alpha_1,\cdot\cdot,x_2,\cdots x_n\right)\;. \tag{13}$$

Then this relation is substituted into the rest of equation (11), and rearrange the terms. The following set of (n-1) equations are obtained as simultaneous Boolean equations.

$$x_j = f_j^{(1)}\left(\alpha_1,\cdot\cdot,x_2,\cdot\cdot,x_{j-1},x_{j+1},\cdot\cdot,x_n,m_1\right)x_j + g_j^{(1)}\left(\alpha_1,\cdot\cdot,x_2,\cdot\cdot,x_{j-1},x_{j+1},\cdot\cdot,x_n,m_1\right) \qquad j = 2\cdots n. \tag{14}$$

Repeat this procedure and finally reach the following equation.

$$x_n = f_n^{(n-1)}\left(\alpha_1,\cdot\cdot,m_1,\cdot\cdot,m_{n-1}\right)x_n + g_n^{(n-1)}\left(\alpha_1,\cdot\cdot,m_1,\cdot\cdot,m_{n-1}\right)\;. \tag{15}$$

The x_n can be expressed only by m_i and fixed Boolean elements α_i as follows,

$$x_n = m_n f_n^{(n-1)}(\alpha_1,\cdot\cdot,m_1,\cdot\cdot,m_{n-1}) + g_n^{(n-1)}(\alpha_1,\cdot\cdot,m_1,\cdot\cdot,m_{n-1}) \quad (16)$$

If we are interested in the rest of x_n, a straight backward substitution leads to the whole solution. For the form of equation(10), the same procedure is also applicable, but the number of indefinite constants increases (m_i-->m_{ik}).

4 Application to reliability engineering

As shown in previous chapter, the solution of simultaneous Boolean equations has many indefinite constants (m_i, m_{ik}), that is, we have almost infinite number of solutions for one set of simultaneous Boolean equations.

It is the reason that the most people did not pay much attention to the fact simultaneous Boolean equations can be solved by algebraic operations[7]. The situation that there exist indefinite constants is similar to that of differential equation. Indefinite constants (C_i) appeared in the solution of differential equation are determined by boundary or initial conditions in solving an actual engineering system.

For Boolean equations, it is necessary to determine terms including m_i or m_{ik} in order to obtain a solution which correctly represent the reliability or availability of an actual engineering system.

In chapter 9 of this paper, it is shown that arbitrary Boolean elements m must be unity or universal set. With this condition, the solution of equation (4) or (9) correctly represents operating state of actual engineering system.

$$m_i \text{ or } m_{ik} = 1 \quad (17)$$

Therefore, the solution of equation (3) becomes,

$$Y = UAD + ABD. \quad (18)$$

The solution of equation (4) becomes,

$$x = f(\alpha_1,\cdot\cdot,\alpha_n) + g(\alpha_1,\cdot\cdot,\alpha_n), \quad (19)$$

instead of equation (8).

5 Engineering systems with loop structure

Two typical examples are taken up as systems with loop structure, and fundamental aspects of loop characteristic and component's role are discussed.

First example is a water supply system as shown in Fig. 2. Pumps suck in water from tanks and pour water into circulated watering facilities. The water flow direction is controlled by check valves. This circulating pipe has branch pipes and supplies water to users.

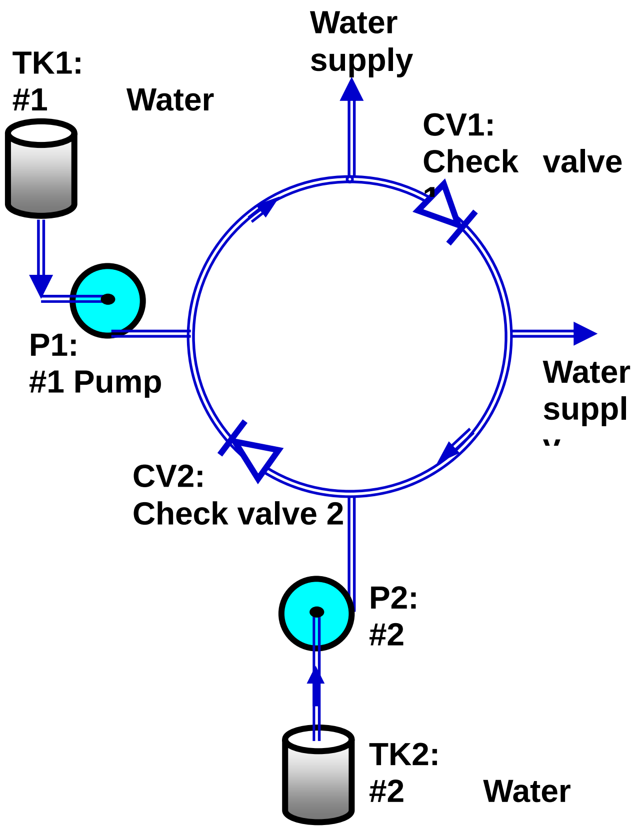

Fig. 2 Water supply system.

In this figure, pumps are considered to be placed outside of the loop structure, because the junction of water pipe has a characteristic of “OR” combination. Water can be supplied to users in case of one pump failure. But, if the both pumps stop, water flow stops at everywhere in the system, because there is no driving force along the circulating pipe (in loop structure).

Next example is an electric power supply system as shown in Fig. 3. Two electric power sources are connected to this loop system. A high voltage and low voltage power sources. A pump, an engine and an electric dynamo are placed in series in this loop system, and produce high voltage electric power. The pump can use low voltage electric power which is converted from high voltage by transformer. The pump sucks in fuel from a fuel tank, and supplies fuel to engine.

In this figure, pump is on the line of loop structure, and fuel tank is considered to be a part of loop structure, because the connection to pump has a characteristic of “AND” combination.

In case of both two external electric power supplies stop, the dynamo can continues supplying high voltage, and both the high and low voltage electric power can be supplied to users.

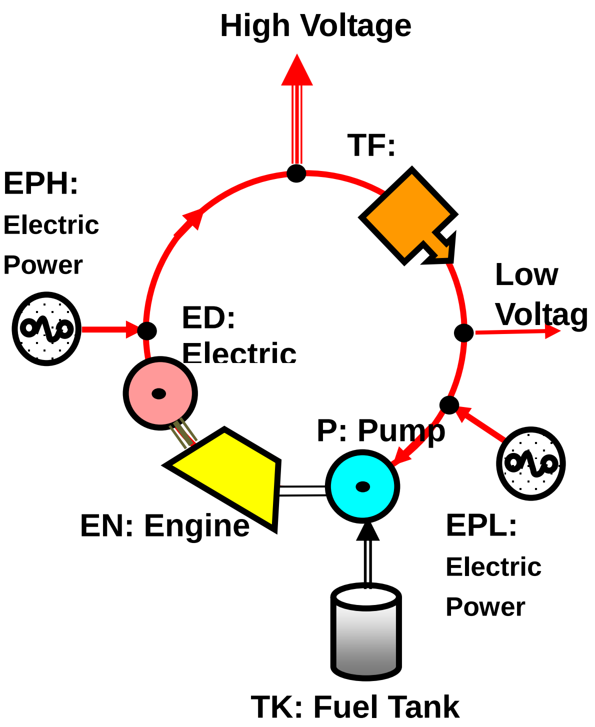

Fig. 3 Electric power supply system.

Systems shown in Figs. 2 and 3 have similar configurations, but the responses of the systems at the failure of outside supports are completely different. The water supply system stops its function, but the electric power supply system continues its function. Along the loop part of electric power supply system, there are four components, and three of them produce outputs as shaft rotating power by engine, electricity by dynamo and fuel flow by pump. These outputs have a kind of driving forces. On the other hand, check vale and transformer do not produce driving force for the purpose of loop operation.

From the above examples, it is seen that a driving force along a loop is required for the continuous operation of a loop without outside support (isolated loop). The example of electric power supply system shows that not all the components along the loop are necessary to produce driving force for the operation of isolated loop.

6 Types of components

The discussions in previous chapter deduce that it is necessary to classify components into different types. The information about component type is used in judging whether loop can continue its operation under isolated condition, or not. There are three types of components; self sustained type, generative type and transmitter type [8].

6.1 Self sustained type component

A self sustained type component (called as "SS-type" in this paper) is a component which can start and continue its operation without any support by other component. The SS-type component does not require support. Examples of this type component are battery, radioisotopic powered generator, accumulator tank, water tank placed at high position, external electric power source and so on.

If a start signal or command is given to a SS-type component, it begins and continues its operation without any support (input) from outside.

6.2 Generative type component

A generative type component (called as "G-type" in this paper) is a component which can produce a driving force for loop operation, in other words, supplies sufficient energy for loop operation. Examples of this type component are engine, electric motor, pump, etc. For the production of its output, a G-type component requires support by other component for its operation.

If there is no support to a G-type component, it can not start its operation even if the G-type component is in sound state. An engine requires fuel supply, an electric motor requires electricity, and a pump requires both water supply and electricity. These G-type components cannot start and cannot continue operation without these supports (inputs).

6.3 Transmitter type component

A transmitter type component (called as "T-type" in this paper) is a component that cannot produce a driving force for loop operation, in other words, does not produce sufficient energy required for loop operation.

Examples of this type component are pipe, wire, an electric transformer, an energy converter, etc. These components just transfer input to output. Pipe and wire transmit exactly the same thing, that is, input and output have same quality, water or electric power or signal. An electric transformer and energy converter modify inputs and produce some different quality of outputs, but they do not give driving force.

For the production of their output, a T-type component also requires input from other component as shown in Figs. 2 and 3. If there is no input to a T-type component, it cannot send out its output even if the T-type component is in sound state.

Components in Fig.1 are classified as follows. Component U is a SS-type. At least one of the components A and B is a G-type component. Component D can be G-type or T-type component.

7 Expression of operating state of a component

In this paper, irreversible change of component's state is assumed. A set representing the sound state of component A in standby condition at time t is expressed by the notation $S_A(t)$. Then, the probability of component A's soundness becomes as follows, for the constant standby failure rate λ_{AO}.

$$\Pr(S_A(t-t_0)) = S_{A0}\cdot\exp(-\lambda_{AS}(t-t_0)), \tag{20}$$

where, S_{A0} is the probability that component A is in sound state at time t_0 (initial time).

The notation $\sigma_{A,t}$ is used for a set representing the event, in which starting signal is given to component A at time t. The notation $D_{A,t}$ is used for a set representing the event, in which component A successfully starts its operation by a given starting signal. Then, $Pr(D_{A,t})$ equals demand probability.

A notation $O_A(t)$ is used for a set representing the operable state of component A after it has been operated during time duration t. For the constant failure rate λ_{AO}, $Pr(O_A(t))$ becomes as follows,

$$\Pr(O_A(t-t_1)) = \exp(-\lambda_{AO}(t-t_1)), \tag{21}$$

where, t_1 is the time in which component A starts its operation.

Now consider the case three different types of components are connected in series as shown in Fig. 4. A SS-type component A and G-type component B are started their operation at time t_1 and t_2, respectively. A T-type component C is considered as a passive component.

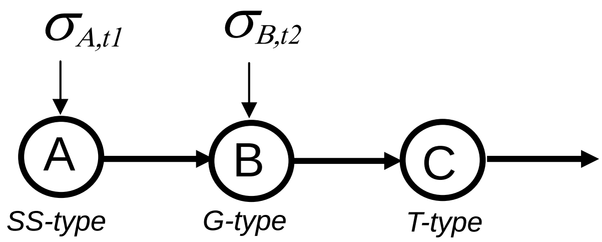

Fig. 4 Three components are connected in series.

A set representing the state in which component A is operating at time t is denoted by $A_O(t)$, where lower suffix "O" means operating state. $A_O(t)$ becomes as follows,

$$A_O(t) = S_A(t_1-t_0)\cdot\sigma_{A,t_1}\cdot D_{A,t_1}\cdot O_A(t-t_1). \tag{22}$$

Component B is a G-type and it requires input for its operation. Then,

$$B_O(t) = A_O(t)\cdot S_B(t_2-t_0)\cdot\sigma_{B,t_2}\cdot D_{B,t_2}\cdot O_B(t-t_2). \tag{23}$$

Component C is a passive component, and it can work if it is in sound state and it receives input.

$$\begin{aligned} C_O(t) &= B_O(t)\cdot S_C(t-t_0) \\ &= S_A(t_1-t_0)\cdot\sigma_{A,t_1}\cdot D_{A,t_1}\cdot O_A(t-t_1) \\ &\quad\cdot S_B(t_2-t_0)\cdot\sigma_{B,t_2}\cdot D_{B,t_2}\cdot O_B(t-t_2)\cdot S_C(t-t_0). \end{aligned} \tag{24}$$

Above expressions are bases in the following discussions, in this paper.

In the above example, start of component B is after the component A, that is, starting signals are given successively in the order of component **link**. The operation of downstream component can be started. The link with this condition is called "**connecting chain**" in this paper.

Next, we consider that more than one starting signals are given to components A and B, as follows. Component A receives $\sigma_{A,t1}$, $\sigma_{A,t4}$, and B receives $\sigma_{B,t2}$, $\sigma_{B,t3}$, in which time sequence is $t_1<t_2<t_3<t_4$.

A set which represents the operating state of component A (SS-type) started by the second starting signal $(\sigma_{A1,t4})$ becomes as follows,

$$A_O^2(t) = \overline{\sigma_{A,t1} D_{A,t1}} \cdot S_A(t4-t0)\cdot \sigma_{A,t4} \cdot D_{A,t4} \cdot O_A(t-t4), \quad (25)$$

where upper suffix "2" of $A_O^2(t)$ means that this set represents the operating state which has been started by the second starting signal. It is assumed that the events $\sigma_{A,t1}$, $\sigma_{B,t2}$, $D_{A,t1}$, $D_{B,t2}$ are mutually independent. In general, the relation $A^i{}_O(t)\ A^j{}_O(t)=\phi(i\neq j)$ holds.

This equation implies that once a component has became in operating state, it does not start again.

For the component B (G-type or T-type), the expression corresponds to equation (23) becomes as follows for $t;\ t3 <t< t4$,

$$\begin{aligned} B_O^2(t) &= A_O^1(t)\cdot\left\{\overline{A_o^1(t2)} + A_o^1(t2)\cdot\overline{\sigma_{B,t2}D_{B,t2}}\right\} \\ &\quad \cdot S_B(t3-t0)\cdot\sigma_{B,t3}\cdot D_{B,t3}\cdot O_B(t-t3) \\ &= A_O^1(t)\cdot\overline{\sigma_{B,t2}D_{B,t2}}\cdot S_B(t3-t0)\cdot\sigma_{B,t3}\cdot D_{B,t3}\cdot O_B(t-t3). \end{aligned} \quad (26)$$

Compare to equation (23), the factor "$\overline{\sigma_{B,t2}D_{B,t2}}$" is added. The reason of this factor is that operating state is started at the second starting chance.

The equation (26) is rewritten into the next form.

$$B_O^2(t) = A_O^1(t)\cdot\overline{\sigma_{B,t2}D_{B,t2}}\cdot B^2(t), \quad (27)$$

where, $B^2(t)$ is an abbreviated expression of $S_B(t3-t0)$ $\cdot\sigma_{B,t3}\cdot D_{B,t3}\cdot O_B(t-t3)$, which means that component B is started at time *t3* (the second starting chance) and becomes in operating condition under the condition of perfect support. But, $B^2(t)$ expresses elementally characteristics of component B itself, and does not express the actual operating state of component B.

After time *t4*, the operating state of component B becomes as follows.

$$B_O(t) = A_O^1(t)\cdot B^1(t) + A_O^1(t)\cdot\overline{\sigma_{B,t2}D_{B,t2}}\cdot B^2(t). \quad (28)$$

8 Operating state of a component with parallel supports

Figure 5 shows a case in which G-type component C is supported by two SS-type components A and B.

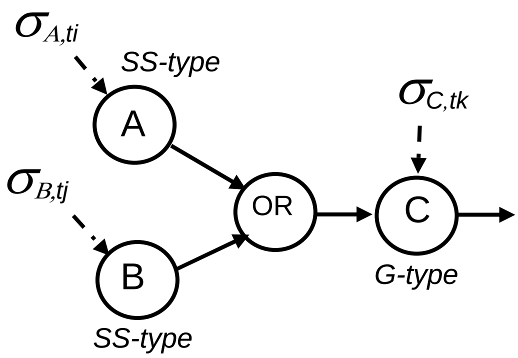

Fig. 5 Supported by two components.

Starting signals to their operations are given at time *ti, tj* and *tk*, respectively.

Now, consider the case in which time sequence for starting signals are given as shown in Table 1.

Table 1 Timing of starting signals given to components

Time	A($\sigma_{A,ti}$)	B($\sigma_{B,tj}$)	C($\sigma_{C,tk}$)
t0			
t1	♦		
t2			♦
t3		♦	
t4			♦
t5	♦		

At time t2, component C can start its operation by the support of component A, but not by component B. There is a "**support gap**" between B and C, that is, component B does not support the start of component C. But, B begins to support the operation of component C after time t3. Then, the operating state of component C at time t;t3<t<t4 becomes as follows.

$$C_O(t) = A_o^1(t)\cdot C^1(t) + B_o^1(t)\cdot A_o^1(t3)\cdot C^1(t).\ ;\quad t3 < t < t4\ . \tag{29}$$

The second term in the equation (29) represents the contribution from the support by B, and this situation is called as "**takeover**" in this paper.

The factor $A_O^1(t3)$ is defined as "**takeover factor**". Even if there is a "support gap" between B and C, the link B-C becomes "connecting chain" by the "takeover" condition.

At time *t4*, the second starting chance is given to component C. Then, operating state of component C after time *t4* becomes,

$$\begin{aligned} C_O(t) &= \left\{A_o^1(t) + B_o^1(t)\cdot A_o^1(t3)\right\}\cdot C^1(t) \\ &\quad + \left\{A_o^1(t) + B_o^1(t)\right\}\cdot\left\{\overline{A_o^1(t2)} + A_o^1(t2)\overline{\sigma_{C,t2}D_{C,t2}}\right\}\cdot C^2(t) \\ &= A_o^1(t)\cdot C^1(t) + B_o^1(t)\cdot A_o^1(t3)\cdot C^1(t) + A_o^1(t)\cdot\overline{\sigma_{C,t2}D_{C,t2}}\cdot C^2(t) \\ &\quad + B_o^1(t)\cdot A_o^1(t2)\cdot\overline{\sigma_{C,t2}D_{C,t2}}\cdot C^2(t) + B_o^1(t)\cdot\overline{A_o^1(t2)}\cdot C^2(t)\ . \end{aligned} \tag{30}$$

The term $\left\{\overline{A_o^1(t2)} + A_o^1(t2)\overline{\sigma_{C,t2}D_{C,t2}}\right\}$, which appear in equation (26), is also seen in this equation. It expresses a set in which component C has not been started at time *t2*.

In the equation (30), the first term is a connecting chain without any interferes from component A. The second term represents a situation that operation of C is supported by component B after time t3. B^1 starts after C^1, then there is a support gap in this link. But, with the takeover effect, this link becomes connecting chain. The third term is a connecting chain. Component C starts at time t4 after failing to start at time t2. The forth term is also a connecting chain, and C^2 is supported by B^1, component C was also failed to start at time t2. The fifth term also represents the situation C^2 is supported by B^1, but component C has not been given a start signal at time t2.

At time *t5*, additional operating state is occurred to component A by the second starting signal. Considering takeover condition (as equation (29)), operating state of component C becomes,

$$\begin{aligned} C_O(t) &= \left\{A_o^1(t) + B_o^1(t)\cdot A_o^1(t3)\right\}\cdot C^1(t) \\ &\quad + \left\{A_o^1(t) + B_o^1(t)\cdot A_o^1(t2)\right\}\cdot\overline{\sigma_{C,t2}D_{C,t2}}\cdot C^2(t) \\ &\quad + B_o^1(t)\cdot\overline{A_o^1(t2)}\cdot C^2(t) \\ &\quad + A_o^2(t)\left\{takeover\ factor\right\}\cdot C^1(t) \\ &\quad + A_o^2(t)\left\{takeover\ factor\right\}\cdot C^2(t). \end{aligned} \tag{31}$$

The forth term becomes,

$$\begin{aligned} &A_o^2(t)\left\{takeover\ factor\right\}\cdot C^1(t) \\ &= A_o^2(t)\left\{A_o^1(t5) + B_o^1(t5)\cdot A_o^1(t3)\right\}\cdot C^1(t) = \phi, \end{aligned} \tag{32}$$

because of $A^1_O(t)A^2_O(t)=\phi$.

The fifth term becomes,

$$\begin{aligned} &A_o^2(t)\left\{takeover\ factor\right\}\cdot C^2(t) \\ &= A_o^2(t)\left\{A_o^1(t5) + B_o^1(t5)\cdot A_o^1(t2)\right\}\cdot\overline{\sigma_{C,t2}D_{C,t2}}\cdot C^2(t) \\ &\quad + A_o^2(t)B_o^1(t5)\cdot\overline{A_o^1(t2)}\cdot C^2(t) \\ &= A_o^2(t)B_o^1(t5)\cdot\overline{A_o^1(t2)}\cdot C^2(t) \end{aligned} \tag{33}$$

Finally, operating state of component C at time *t>t5* becomes as follows,

$$\begin{aligned} C_O(t) &= \left\{A_o^1(t) + B_o^1(t)\cdot A_o^1(t3)\right\}\cdot C^1(t) \\ &\quad + \left\{A_o^1(t) + B_o^1(t)\cdot A_o^1(t3)\right\}\cdot\overline{\sigma_{C,t2}D_{C,t2}}\cdot C^2(t) \\ &\quad + B_o^1(t)\cdot\overline{A_o^1(t2)}\cdot C^2(t) \\ &\quad + A_o^2(t)\cdot B_o^1(t5)\cdot\overline{A_o^1(t2)}\cdot C^2(t). \end{aligned} \tag{34}$$

The last term is an additional term. It represents the situation that A^2 supports C^2 after time t5, by the takeover effect. In this case, C^2 is purely started by the support of B^1 not by the support of A^1.

9 Analysis of loop structure

9.1 Analysis procedure by Boolean equation

Any system performance can be expressed by Boolean relations. The analysis is performed by the following steps.

1) Construct Boolean equation(s), which expresses loop structure(s) with unknown Boolean element(s).
2) Eliminate the terms, which does not become a "connecting chain".
3) Eliminate the unknown element(s) by the rule presented in chapter 2 (equations (8) and (10)).
4) Classify the terms as non-loop or loop terms on the bases of arbitrary Boolean elements *m*.

5) Obtain concrete expressions of connecting chains for non-loop terms with the takeover condition, if necessary.
6) Eliminate the terms, which do not contain G-type component from circular link terms (loop structure).
7) Find out the “support gap” in the remaining circular links, and eliminate "non-operable" links.
8) Give concrete expressions for circular link terms. At least one “takeover” condition exists along the circular links.

Previous study [8] shows that after the establishment of a loop operation, additional inputs from the outside of loop structure have no influence to the operating states of the loop.

9.2 Sample system

A sample system is taken up as shown in Fig. 6 and analysis is performed by the proposed method. By this analysis, it becomes clear the procedure for solving a typical loop structure with multiple starting signal for each component.

The system is a general and rather abstract system which has one loop structure with two external supports.

Components A and B are allied in parallel position, and they are connected to component E (Electric generator is suggested.). Components E and P (Fuel pump is suggested.) make a loop structure. X is an output of component P and it includes the operating state of loop structure. Y is an output of component E and it is system output (It suggests electric power supply to users.).

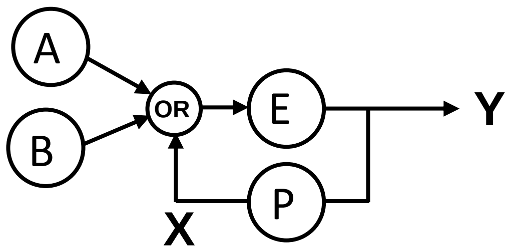

Fig. 6 A sample system.

Multiple starting signals are given to components for their operations as shown in Table 2.

Table 2 Timing of starting signals given to components.

Time	A	B	E	P
t0				
t1	♦			
t2			♦	
t3				♦
t4		♦		
t5			♦	
t6				♦
t7	♦			

9.3 Analysis of the sample system

First, the system state is expressed by the next Boolean equations which represents the system relation at the time $t>t_7$. At this time t, starting signals are given to all the components. Then the Boolean equation contains all the possible states of system operation.

$$X = (A_O + B_O + X)\cdot E_O \cdot P_O,$$
$$Y = (A_O + B_O + X)\cdot E_O. \qquad (35)$$

Boolean element, A_O $(B_O, E_O.P_O)$ represents operating state of component A (B,E,P). “Operating state” is rather conceptual at this stage. Concrete expressions of elements E_O, P_O are identified after solving the equation (35).

The Boolean relation (35) is constructed with the condition that G or T-type component requires an input for its operation.

As seen in Table 2, multiple starting signals are given to the components A, E, P. Then, the elements A_O, E_O, P_O are divided into two parts corresponding to the different starting time. For example, $A_O = A_O^1 + A_O^2$, where A_O^1 means operating state of component A started at the first chance. The relations $A_O^i A_O^j = \phi$ $(i \neq j)$ hold.

The equation (35) is rewritten,

$$X = (A_O^1 + A_O^2 + B_O + X)\cdot\left(E_O^1 + E_O^2\right)\cdot\left(P_O^1 + P_O^2\right). \qquad (36)$$

From equation (36), following equation is obtained,

$$\begin{aligned} X = {} & A_O^1E_O^1P_O^1 + A_O^1E_O^1P_O^2 + A_O^1E_O^2P_O^1 + A_O^1E_O^2P_O^2 \\ & + A_O^2E_O^1P_O^1 + A_O^2E_O^1P_O^2 + A_O^2E_O^2P_O^1 + A_O^2E_O^2P_O^2 \\ & + B_OE_O^1P_O^1 + B_OE_O^1P_O^2 + B_OE_O^2P_O^1 + B_OE_O^2P_O^2 \\ & + (E_O^1P_O^1 + E_O^1P_O^2 + E_O^2P_O^1 + E_O^2P_O^2)\cdot X. \end{aligned} \qquad (37)$$

Eliminate the terms which never become connecting chains. For example, in term $A_O^2E_O^1P_O$, E_O^1 starts at time $t2$, but A_O^2 starts later time $t7$. There is a "support gap" in the link A_O^2-E_O^1 and there is no "takeover" condition between them.

$$\begin{aligned} X = {} & A_O^1E_O^1P_O^1 + A_O^1E_O^1P_O^2 + A_O^1E_O^2P_O^1 + A_O^1E_O^2P_O^2 \\ & + A_O^2E_O^2P_O^1 + A_O^2E_O^2P_O^2 + B_OE_O^1P_O^1 + B_OE_O^1P_O^2 + B_OE_O^2P_O^2 \\ & + E_O^1P_O^1X + E_O^1P_O^2X + E_O^2P_O^2X. \end{aligned} \tag{38}$$

Eliminate the X in right hand side, by the rule presented in chapter 2 (equation (10)).

$$\begin{aligned} X = {} & A_O^1E_O^1P_O^1 + A_O^1E_O^1P_O^2 + A_O^1E_O^2P_O^1 + A_O^1E_O^2P_O^2 \\ & + A_O^2E_O^2P_O^1 + A_O^2E_O^2P_O^2 + B_OE_O^1P_O^1 + B_OE_O^1P_O^2 + B_OE_O^2P_O^2 \\ & + m_1E_O^1P_O^1 + m_2E_O^1P_O^2 + m_3E_O^2P_O^2. \end{aligned} \tag{39}$$

In the equation (39), terms from $A_O^1E_O^1P_O^1$ to $B_OE_O^2P_O^2$ represent non-loop operations and the last three terms ($m_1E_O^1P_O^1$, $m_2E_O^1P_O^2$, $m_3E_O^2P_O^2$) represent loop operations. These situations are confirmed by the Fig. 6.

As shown in 9.3.2, $m_1=m_2=m_3=1$, then they are replaced by m, which becomes an indicator that a term represents a loop structure.

$$\begin{aligned} X = {} & A_O^1E_O^1P_O^1 + A_O^1E_O^1P_O^2 + A_O^1E_O^2P_O^2 + A_O^2E_O^2P_O^1 \\ & + A_O^2E_O^2P_O^2 + B_OE_O^1P_O^1 + B_OE_O^1P_O^2 + B_OE_O^2P_O^2 \\ & + mE_O^1P_O^1 + mE_O^1P_O^2 + mE_O^2P_O^2. \end{aligned} \tag{40}$$

9.3.1 Concrete expressions of non-loop terms

Next, obtain concrete expressions of connecting chains for non-loop terms. The links between components are seen from terms in equation (40). Concrete expressions are derived by previous chapter's results and by considering the timing of starting times and takeover conditions.

Term $A_O^1E_O^1P_O^1$, linking order of components A-E-P and starting order of component's operations are consistent. Then, connecting chain becomes,

$$A_O^1E_O^1P_O^1 = A_o^1(t)\cdot E^1(t)\cdot P^1(t) \;\; ; \;\; t_3 < t\,. \tag{41}$$

Term $A_O^1E_O^1P_O^2$, the operating state of component E supported by A^1 becomes as follows.

$$E_o^1(t) = A_o^1(t)E^1(t). \tag{42}$$

The start of component P is given at the second chance, then the operating state of component P supported by E^1 becomes as follows.

$$P_o^2(t) = E_O^1(t)\cdot\left\{\overline{A_o^1(t_3)E^1(t_3)} + A_o^1(t_3)E^1(t_3)\cdot\overline{\sigma_{P,t3}D_{P,t3}}\right\}\cdot P^2(t) \tag{43}$$

The bracket means a set in which component P has not been started at time t_3. This form is appeared in equations (26) and (30).

From equation (43), following equation is obtained,

$$P_o^2(t) = A_o^1(t)E^1(t)\cdot\overline{\sigma_{P,t3}D_{P,t3}}\cdot P^2(t). \tag{44}$$

Therefore,

$$A_O^1E_O^1P_O^2 = A_o^1(t)E^1(t)\cdot\overline{\sigma_{P,t3}D_{P,t3}}\cdot P^2(t) \;\; ; \;\; t_6 < t\,. \tag{45}$$

Term $A_O^1E_O^2P_O^2$, the operating state of component E started at the second starting chance becomes,

$$E_o^2(t) = A_o^1(t)\cdot\overline{\sigma_{E,t2}D_{E,t2}}\cdot E^2(t). \tag{46}$$

Then, the operating state of P supported by E_O^2 becomes as follows.

$$\begin{aligned} P_o^2(t) &= E_o^2(t)\cdot\left\{\overline{A_o^1(t_3)E^1(t_3)} + A_o^1(t_3)E^1(t_3)\cdot\overline{\sigma_{P,t3}D_{P,t3}}\right\}\cdot P^2(t) \\ &= A_o^1(t)\cdot\overline{\sigma_{E,t2}D_{E,t2}}\cdot E^2(t)P^2(t). \end{aligned} \tag{47}$$

Finally,

$$A_O^1E_O^2P_O^2 = A_o^1(t)\cdot\overline{\sigma_{E,t2}D_{E,t2}}\cdot E^2(t)P^2(t) \;\; ; \;\; t_6 < t\,. \tag{48}$$

Term $A_O^2E_O^2P_O^1$, the operating state of A is started at the second starting chance, then,

$$A_o^2 = A_O^2(t) = \overline{\sigma_{A,t1}D_{A,t1}}\cdot S_A(t_7 - t_0)\cdot\sigma_{A,t7}\cdot D_{A,t7}\cdot O_A(t - t_7). \tag{49}$$

This is equivalent to equation (25).

Link A_O^2-E_O^2 has a "support gap", so consider the takeover condition.

$$E_o^2(t) = A_o^2(t)\{takeoverfactor\}E^2(t). \tag{50}$$

From the equation (30), takeover factor can be determined.

$$\begin{aligned} E_o^2(t) &= A_o^2(t)\left\{\begin{array}{c}\left\{A_o^1(t_7) + B_o(t_7)A_o^1(t_2)\right\}\cdot\overline{\sigma_{E,t2}D_{E,t2}} \\ + B_o^1(t_7)\cdot\overline{A_o^1(t_2)}\end{array}\right\}E^2(t) \\ &= A_o^2(t)B_o^1(t_7)\cdot\overline{A_o^1(t_2)}E^2(t). \end{aligned} \tag{51}$$

Next link E_O^2-P_O^1 has also a "support gap", so consider the takeover condition again.

$$P_o^1(t) = E_o^2(t)\{takeoverfactor\} P^1(t). \quad (52)$$

Therefore,

$$P_o^1(t) = E_o^2(t)\left\{A_O^1(t_7)E^1(t_7) + B(t_7)A_O^1(t_4)E^1(t_4)\right\} P^1(t) = \phi. \quad (53)$$

$$A_O{}^2E_O{}^2P_O{}^1 = \phi. \quad (54)$$

In the same way, rests of the terms are determined as follows,

$$\begin{aligned} A_O{}^2E_O{}^2P_O{}^2 &= A_o^2(t)B_o(t_7)\cdot\overline{A_o^1(t_2)}E^2(t)P^2(t),\\ B_OE_O{}^1P_O{}^1 &= B_o(t)\cdot A_o^1(t_4)E^1(t)P^1(t),\\ B_OE_O{}^1P_O{}^2 &= B_o(t)\cdot A_o^1(t_4)E^1(t)\overline{\sigma_{P,t3}D_{P,t3}}P^2(t),\\ B_OE_O{}^2P_O{}^2 &= B_o(t)\cdot\left\{\overline{A_o^1(t_2)} + A_o^1(t_4)\overline{\sigma_{E,t2}D_{E,t2}}\right\}\cdot E^2(t)P^2(t). \end{aligned} \quad (55)$$

9.3.2 Operating states of loop structure

The terms which contain an arbitrary element m are considered as circular links and they represent operating states of loop structures.

In non-loop links, starting component is always SS-type component, and its operating state can be determined by the form of equation (22) or (25).

But, in circular links, there is no SS-type component. Operating state of G- or T-type component cannot be determined without determining input event.

Consider a circular link C1-C2-…CN-C1, that is, there are N components in the link.

Now, let us try to calculate operating state of the term "$C1^i$-$C2^j$-…CN^n" as follows.

Select one component for the starting point, for example C1. Make an assumption that $C1^i(t)$ is the operating state of $C1_O{}^i$, because we cannot determine input to C1. Definition $C1^i(t)$ is given by the equation (27) and upper suffix i means i-th starting chance.

Then along the link direction, successively obtain operating state till reach the C1 itself again. The process to obtain the operating states of successive components is the same to the one presented in the previous section (for non-loop terms).

Operating sate of the last component ($C1^i$ in this case) becomes "Operating state of loop structure". It has been confirmed by comparing to the result obtained by the engineering consideration [8].

Above procedure gives "Operating state of loop structure", and states of loop operation is equal at everywhere in the loop structure [8]. Then, operating states of components placed in an isolated loop structure are determined as,

$$C1_o^i(t) = C2_o^j(t) = \cdots = CN_o^n(t). \quad (56)$$

Therefore, m is determined as unity.

$$\begin{aligned} &mC1_o^i C2_o^j \cdots CN_o^n \\ &\quad = C1_o^i(t)C2_o^j(t)\cdots CN_o^n(t) = CK_o^k(t), \end{aligned} \quad (57)$$

where, CK is any component in circular link.

In the followings, circular link is E-P.

Term $mE_O{}^1P_O{}^1$, the circular link is $E_O{}^1$-$P_O{}^1$-$E_O{}^1$. There is no "support gap" between $E_O{}^1$-$P_O{}^1$, but there is "support gap" between $P_O{}^1$-$E_O{}^1$, and the "Takeover factor" is $A_O{}^1(t_3)$. Therefore,

$$mE^1P^1 = A_o^1(t_3)\cdot E^1(t)P^1(t) \;\; ; \;\; t_3 < t. \quad (58)$$

Term $mE_O{}^1P_O{}^2$, the circular link is $E_O{}^1$-$P_O{}^2$-$E_O{}^1$. There is no "support gap" between $E_O{}^1$-$P_O{}^2$, but $P_O{}^2$ is operating state started at the second starting chance.

$$\begin{aligned} P_o^2 &= E^1(t)\cdot\left\{\overline{E^1(t)} + E^1(t)\cdot\overline{\sigma_{P,t3}D_{P,t3}}\right\}\cdot P^2(t)\\ &= E^1(t)\cdot\overline{\sigma_{P,t3}D_{P,t3}}\; P^2(t). \end{aligned} \quad (59)$$

There is "support gap" between $P_O{}^2$-$E_O{}^1$. The "Takeover factor" is $A^1{}_O(t_6)$+$B^1{}_O(t_6)A^1{}_O(t_4)$ as the operating state of $E_O{}^1(t)$ is deduced from equation (29). Therefore,

$$mE_O{}^1P_O{}^2 = \left\{A_o^1(t_6) + B_o(t_6)A_o^1(t_4)\right\}\cdot E^1(t)\overline{\sigma_{P,t3}D_{P,t3}}P^2(t) \;\; ; \;\; t_6 < t. \quad (60)$$

Term $mE_O{}^2P_O{}^2$, the circular link is $E_O{}^2$-$P_O{}^2$-$E_O{}^2$. There is no "support gap" between $E_O{}^2$-$P_O{}^2$. The $P_O{}^2$

is operating state started at the second starting chance.

$$P_O^2 = E^2(t) \cdot \left\{ \overline{E^1(t)} + E^1(t) \cdot \overline{\sigma_{P,t3} D_{P,t3}} \right\} \cdot P^2(t) \\ = E^2(t) \cdot P^2(t). \quad (61)$$

There is "support gap" between $P_O{}^2$-$E_O{}^2$. Operating state of component E started at the second starting chance is also deduced from equation (30). Then, "takeover factor" becomes,

$$\left\{ A_o^1(t_6) + B_o(t_6) A_o^1(t_2) \right\} \cdot \overline{\sigma_{E,t2} D_{E,t2}} + B_o^1(t_6) \cdot \overline{A_o^1(t_2)} \ . \quad (62)$$

The term $mE_O{}^2P_O{}^2$ is determined as,

$$mE_O{}^2P_O{}^2 = \\ \left[\left\{ A_o^1(t_6) + B_o(t_6) A_o^1(t_2) \right\} \cdot \overline{\sigma_{E,t2} D_{E,t2}} + B_o^1(t_6) \cdot \overline{A_o^1(t_2)} \right] \cdot E^2(t) P^2(t) \\ ; \ t_6 < t. \quad (63)$$

Output of component P is represented by X in Fig. 6. The X becomes as follows by getting together the above results and rearranging.

$$X = P_O = \left\{ A_o^1(t) + B_o(t) \cdot A_o^1(t_4) \right\} \cdot E^1(t) \cdot \left\{ P^1(t) + \overline{\sigma_{P,t3} D_{P,t3}} \cdot P^2(t) \right\} \\ + \left\{ A_o^1(t) + B_o(t) A_o^1(t_4) \right\} \cdot \overline{\sigma_{E,t2} D_{E,t2}} \cdot E^2(t) P^2(t) \\ + B_o(t) \cdot \overline{A_o^1(t_2)} \cdot E^2(t) P^2(t) + A_o^2(t) B_o(t_7) \cdot \overline{A_o^1(t_2)} E^2(t) P^2(t) \\ + A_o^1(t_3) \cdot E^1(t) P^1(t) + \left\{ A_o^1(t_6) + B_o(t_6) A_o^1(t_4) \right\} \cdot E^1(t) \overline{\sigma_{P,t3} D_{P,t3}} P^2(t) \\ + \left[\left\{ A_o^1(t_6) + B_o(t_6) A_o^1(t_2) \right\} \cdot \overline{\sigma_{E,t2} D_{E,t2}} + B_o(t_6) \cdot \overline{A_o^1(t_2)} \right] \cdot E^2(t) P^2(t). \quad (64)$$

9.3.3 Output of sample system

Output of the sample system is represented by Y, which is shown in Fig. 6. Y consists of direct connecting chains from components A and B (non-loop operating states) and operating states of loop structure.

Output of component C in Fig. 5 is exactly the same to the output of component E in Fig. 6, for non-loop connecting chains.

As states of loop operation is equal at everywhere in the loop structure[8], the contribution to Y from loop operation is obtained as loop operating states at X, that is, the last three terms in equation (64).

The output of the sample system becomes,

$$Y = E_O = \left\{ A_o^1(t) + B_o(t) \cdot A_o^1(t_4) \right\} \cdot E^1(t) \\ + \left\{ A_o^1(t) + B_o(t) A_o^1(t_4) \right\} \cdot \overline{\sigma_{E,t2} D_{E,t2}} \cdot E^2(t) \\ + B_o(t) \cdot \overline{A_o^1(t_2)} \cdot E^2(t) + A_o^2(t) B_o(t_7) \cdot \overline{A_o^1(t_2)} E^2(t) \\ + A_o^1(t_3) \cdot E^1(t) P^1(t) + \left\{ A_o^1(t_6) + B_o(t_6) A_o^1(t_4) \right\} \cdot E^1(t) \overline{\sigma_{P,t3} D_{P,t3}} P^2(t) \\ + \left[\left\{ A_o^1(t_6) + B_o(t_6) A_o^1(t_2) \right\} \cdot \overline{\sigma_{E,t2} D_{E,t2}} + B_o^1(t_6) \cdot \overline{A_o^1(t_2)} \right] \cdot E^2(t) P^2(t). \quad (65)$$

The above results, $E_O(t)$, $P_O(t)$ were confirmed by comparing the results obtained by a step by step analysis based on the engineering considerations[9].

10 Conclusions

The procedure for solving Boolean equations with unknown element(s) has been derived by basic Boolean rules [7].

Two typical examples of engineering systems were taken up as having a loop structure and their system behavior was examined. It was revealed that components are necessary to be classified into three different types. Self sustained type (SS-type), Generative type (G-type) and Transmitter type (T-type).

Time-dependent expressions of component's operating states were given and the expressions were used to examine fundamental behavior of G- and T-type components. It was pointed out that "takeover" was observed in case of parallel supports of two SS-type components to one G-type component.

A procedure to identify operating states of loop structure has been proposed. The procedure deduced an important finding that an arbitrary Boolean element m is set to be unity for solving Boolean equation with unknown elements. Operating state of loop structure can be obtained by Boolean algebraic procedure.

The procedure is applicable to the condition that components can start at any time in system operational sequence, and each component has multiple chances to be started.

An example system was taken up and solved by this procedure for the explanation of procedure. The analysis result was confirmed by a step by step analysis based on the engineering considerations.

The procedure shown in this paper is very useful in evaluating engineering systems which have logical loop structure(s), and also useful in effectively designing high reliable systems.

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